

# Hydrodynamics beyond local equilibrium: application to electron gas

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Generalized hydrodynamic theory, which does not rest on the requirement of a local equilibrium, is derived in the long-wave limit of a kinetic equation. The theory bridges the whole frequency range between the quasistatic (Navier-Stokes) hydrodynamics and the high frequency (Vlasov) collisionless limit. In addition to pressure and velocity the theory includes new macroscopic tensor variables. In a linear approximation these variables describe an effective shear stress of a liquid and the generalized hydrodynamics recovers the Maxwellian theory of highly viscous fluids - the media behaving as solids on a short time scale, but as viscous fluids on long time intervals. It is shown that the generalized hydrodynamics can be applied to the Landau theory of Fermi liquid. Illustrative results for collective modes in confined systems are given, which show that nonequilibrium effects qualitatively change the collective dynamics in comparison with the predictions of the heuristic Bloch's hydrodynamics. Earlier improvements of the Bloch theory are critically reconsidered.

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## I. INTRODUCTION

The idea to apply a macroscopic hydrodynamical description to the collective dynamics of the inhomogeneous many-electron systems has been first suggested in a famous work by Bloch in 1933.<sup>1</sup> In this paper he introduced a hydrodynamic theory of a degenerate Fermi gas as a simplest phenomenological extension of the Thomas-Fermi model to dynamical regime. Since application of nonequilibrium many-body theory to inhomogeneous systems is extremely complex, the Bloch's hydrodynamic theory (BHT) remains popular and is in use until now. Since 1933 the BHT has been applied to variety of kinetic problems,<sup>2-8</sup> albeit due to the heuristic nature of the BHT, its relation to the kinetic theory and the range of applicability remained unclear.

A common sense of hydrodynamical theory presumes an existence of a local statistical equilibrium in every point of space - the condition which allows to express the kinetic properties of a many-body system in simple terms of pressure, density and temperature distributions.<sup>9</sup> This condition is, however, rarely met in electron gas where the electron-electron collisions commonly play a minor role. This circumstance is ignored in the BHT. Although the applications of the Bloch's hydrodynamical theory<sup>2-8,10</sup> have shown its usefulness, its failures are also well known, in particular a prediction of a wrong plasmon dispersion.<sup>4-7,10-12</sup>

Is it possible to derive a macroscopic hydrodynamics-like theory which would be valid beyond the condition of a local equilibrium? What is the role of collisions and how is the static (i.e. collision dominated) limit as well as the high frequency (collisionless) limit recovered? These are the questions we address in the present paper. In the short publication<sup>13</sup> we showed that such hydrodynamics existed. In the paper Ref. 13, for the sake of simplicity we carried out the derivation of the hydrodynamics equa-

tions only in one dimensional case. We also restricted the theory to the second order in the long-wave limit of a kinetic equation, which left out the thermal conductivity and consequently the thermal equilibration processes.

In the present paper we give a new general derivation and extend the theory to include the heat transport. We show that the generalized hydrodynamics can be based on Landau theory of Fermi liquid and relate macroscopic parameters in hydrodynamics equations (i.e. shear and bulk moduli), to microscopic Landau parameters. We also present transparent analytical examples of the calculation of collective modes in confined systems of different dimensionality.

Let us first consider the mathematical structure of BHT and the formal microscopic restrictions of its applicability. The Bloch's original idea was to use the set of hydrodynamics equations of an ideal charged liquid to describe the dynamics of a Fermi gas. Only the macroscopic variables - electron density  $n(\mathbf{r}, t)$ , velocity  $\mathbf{v}(\mathbf{r}, t)$ , pressure  $P$ , and electrostatic potential  $\varphi(\mathbf{r}, t)$ , enter the equations:

$$D_t n + n \nabla \mathbf{v} = 0, \quad (1)$$

$$m n D_t \mathbf{v} + \nabla P - e n \nabla \varphi = 0, \quad (2)$$

$$\nabla^2 \varphi = 4\pi e n - 4\pi \rho_{ext}, \quad (3)$$

where  $\rho_{ext}$  is an external charge and  $D_t = \partial_t + \mathbf{v} \nabla$  is a substantial derivative. This set of equations (continuity equation (1), Euler (2) and Poisson (3) equations) becomes complete when the equation of state is added. In the original paper<sup>1</sup> Bloch identified  $P$  with the kinetic pressure of a degenerate Fermi gas:

$$P = \frac{1}{5m} (3\pi^2)^{2/3} n^{5/3} \quad (4)$$

Further improvements of BHT addressed only the equation of state (inclusion of exchange, correlation and quantum gradient corrections).<sup>8</sup>

Sometimes BHT is viewed as an approximate extension of the density functional theory to the dynamic regime.<sup>8</sup> However, as we have mentioned above, the hydrodynamics theory (1)-(4) exhibits an inconsistency which has been reflected in many papers and textbooks.<sup>4-7,10-12</sup> The system of equations (1)-(4) gives the wrong velocity coefficient  $v_0^2$  in the plasmon dispersion law  $\omega^2 = \omega_p^2 + v_0^2 q^2$ . Instead of the correct result  $\frac{3}{5}v_F^2$  ( $v_F$  is the Fermi velocity) of the linear response theory (RPA) the BHT leads to the value  $v_0^2 = \frac{1}{3}v_F^2$ . This inconsistency does not depend on a degree of degeneracy of an electron gas and is the general property of the hydrodynamics theory which is based on the equilibrium equation of state. For example, at arbitrary degeneracy the result of hydrodynamics is  $v_0^2 = v_s^2$ , where  $v_s$  is the velocity of sound, whereas in the linear response theory  $v_0^2$  equals to the mean square of the particle velocity  $<v_p^2>$ .<sup>12</sup>

It has been realized<sup>10</sup> that this discrepancy originates from the assumption of a local equilibrium, which underlies the common hydrodynamic theory.<sup>9</sup> The assumption allows to reduce the kinetic equation for the distribution function  $f_{\mathbf{p}}(\mathbf{r}, t)$

$$\partial_t f_{\mathbf{p}} + \frac{\mathbf{p}}{m} \nabla f_{\mathbf{p}} + e \nabla \varphi \frac{\partial f_{\mathbf{p}}}{\partial \mathbf{p}} = I_{\mathbf{p}}[f_{\mathbf{p}}] \quad (5)$$

to a set of equations for macroscopic variables  $n(\mathbf{r}, t), \mathbf{v}(\mathbf{r}, t)$  and, in general case, temperature  $T(\mathbf{r}, t)$ . The requirement of the local equilibrium is fulfilled if the characteristic time of the process  $t_{pr} \sim 1/\omega$  is much longer than the inverse collision frequency  $1/\nu_c$  and the typical length of the inhomogeneity  $L$  is greater than the mean free path  $l \sim u/\nu_c$  ( $u$  is the average particle velocity)<sup>14</sup>

$$\omega/\nu_c \ll 1, \quad u/L\nu_c \ll 1. \quad (6)$$

In a zero order with respect to parameters (6) Eq. (5) reduces to  $I_{\mathbf{p}}[f_{\mathbf{p}}] = 0$ , which means that the distribution function  $f_{\mathbf{p}}$  and, consequently, the equation of state has a locally equilibrium form. As a result, the equations for the first three moments of the distribution function i.e. density  $n$ , current  $\mathbf{j} = n\mathbf{v}$  and stress tensor  $P_{ij} = \delta_{ij}P$  (which is diagonal in the local equilibrium) constitute the closed set of hydrodynamics equations for an ideal liquid.

It is clear that from the microscopic point of view BHT cannot be a consistent theory since it extends the collision dominated hydrodynamics ( $\nu_c \rightarrow \infty$ ) to the electron gas where the collisionless (Vlasov) limit ( $\nu_c \rightarrow 0$ ) is most common. Due to the high frequency of the plasma waves at least the first of inequalities (6) is strongly violated and the tensor structure of  $P_{ij}$  as well as that of higher moments becomes important. In a degenerate case these tensors describe deformation of the Fermi sphere - the effect completely ignored in the BHT. It has been shown recently<sup>15</sup> that within a linear response theory this effect leads to an effective shear modulus of a liquid.

In this paper we present a generalized hydrodynamics which remains valid far beyond the local equilibrium,

when both conditions (6) are strongly violated. The theory is restricted to the long-wave limit of the kinetic equation and requires inclusion of the tensors of higher moments. In Sec. II we derive a general hierarchy of equations for the moments of the distribution function and introduce a regular procedure of truncation of the infinite chain of equations, which does not require the assumption of a local equilibrium. We also discuss the physical meaning of restrictions which make possible the truncation. In Sec. III we construct a hydrodynamic theory of the second order in the long wavelength expansion and show a relation of the generalized hydrodynamics to the theory of elasticity and to the theory of highly viscous fluids. Sec. IV is devoted to the generalized hydrodynamics of a charged Fermi-liquid on the basis of Landau theory. We find the relationship between macroscopic hydrodynamical parameters and microscopic Fermi-liquid parameters. Sec. V contains examples of application of the generalized hydrodynamics to collective oscillations of confined systems. The purpose of this section is purely illustrative. To show clearly the nonequilibrium effects we select an analytically solvable model of a charged Fermi liquid in a harmonic potential of different dimensionality. We show that deviations from the local equilibrium strongly shift the excitation frequencies and substantially change the structure of collective modes. In Sec. VI we extend the theory to the fourth order in the long wavelength expansion. It is shown that in the low frequency limit the fourth-order theory transforms into the set of equations of a classical hydrodynamics, which includes viscosity and a heat transport contribution. In Sec. VII we summarize our results.

## II. THE HIERARCHY OF EQUATIONS FOR THE MOMENTS

The kinetic equation (5) can be transformed in an infinite chain of equations for the moments of the distribution function.<sup>14</sup> The zeroth and the first moment are the particle density  $n(\mathbf{r}, t)$  and the current density  $\mathbf{j}(\mathbf{r}, t)$ :

$$n(\mathbf{r}, t) = \sum_{\mathbf{p}} f_{\mathbf{p}}(\mathbf{r}, t), \quad \mathbf{j}(\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{\mathbf{p}}{m} f_{\mathbf{p}}(\mathbf{r}, t) \quad (7)$$

The velocity field can be defined in a usual way as  $\mathbf{v} = \mathbf{j}/n$ . In general, the moment of the  $k$ th order is a tensor of  $k$ th rank

$$\mathbf{M}^{(k)} = M_{i_1 \dots i_k}^{(k)} = \frac{1}{m^{k-1}} \sum_{\mathbf{p}} p_{i_1} \dots p_{i_k} f_{\mathbf{p}}, \quad k > 1 \quad (8)$$

It is a common practice to separate the macroscopic motion of the liquid from the relative motion of particles via transformation to the Lagrange comoving frame. In this frame the macroscopic velocity is equal to zero and the particle density  $n(\mathbf{r}, t)$  is the same as in the laboratory frame. The transformed distribution function  $f_{\mathbf{p}}^L(\mathbf{r}, t)$

is related to the distribution function in the laboratory frame  $f_{\mathbf{p}}(\mathbf{r}, t)$  as follows

$$f_{\mathbf{p}}^L = f_{\mathbf{p}+\mathbf{m}\mathbf{v}}$$

and apparently has the properties

$$\begin{aligned} \sum_{\mathbf{p}} f_{\mathbf{p}}^L(\mathbf{r}, t) &= \sum_{\mathbf{p}} f_{\mathbf{p}}(\mathbf{r}, t) = n(\mathbf{r}, t), \\ \sum_{\mathbf{p}} \frac{\mathbf{p}}{m} f_{\mathbf{p}}^L(\mathbf{r}, t) &= \sum_{\mathbf{p}} \frac{\mathbf{p}}{m} f_{\mathbf{p}}(\mathbf{r}, t) - \mathbf{v}(\mathbf{r}, t)n(\mathbf{r}, t) = 0. \end{aligned} \quad (9)$$

Eqs. (9) and (10) can be considered as a definition of the comoving frame.

The higher moments of the transformed distribution function are defined similar to (8):

$$\mathbf{L}^{(k)} = L_{i_1 \dots i_k}^{(k)} = \frac{1}{m^{k-1}} \sum_{\mathbf{p}} p_{i_1} \dots p_{i_k} f_{\mathbf{p}}^L, \quad k > 1 \quad (11)$$

Equations for moments  $L_{i_1 \dots i_k}^{(k)}$  can be easily derived from the kinetic equation for the distribution function  $f_{\mathbf{p}}^L$ :

$$\begin{aligned} D_t f_{\mathbf{p}}^L + \frac{\mathbf{p}}{m} \nabla f_{\mathbf{p}}^L - (\mathbf{p} \nabla) \mathbf{v} \frac{\partial f_{\mathbf{p}}^L}{\partial \mathbf{p}} \\ - (m D_t \mathbf{v} - e \nabla \varphi) \frac{\partial f_{\mathbf{p}}^L}{\partial \mathbf{p}} = I_{\mathbf{p}}^L [f_{\mathbf{p}}^L], \end{aligned} \quad (12)$$

where  $D_t = \partial_t + \mathbf{v} \nabla$  is a substantial derivative. The zeroth and the first moments of this equation give respectively the continuity equation and the equation for the velocity  $\mathbf{v}$ :

$$D_t n + n \nabla_i v_i = 0 \quad (13)$$

$$m n D_t v_i + \nabla_j L_{ij}^{(2)} - e n \nabla_i \varphi = 0 \quad (14)$$

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$$\begin{aligned} \frac{1}{m^{k-1}} \sum_{\mathbf{p}} p_{i_1} \dots p_{i_k} p_j \frac{\partial f_{\mathbf{p}}^L}{\partial p_i} \nabla_j v_i &= - \frac{1}{m^{k-1}} \sum_{\mathbf{p}} f_{\mathbf{p}}^L \frac{\partial}{\partial p_i} p_{i_1} \dots p_{i_k} p_j \nabla_j v_i \\ &= \left[ \delta_{i i_1} L_{i_2 \dots i_k j}^{(k)} + \delta_{i i_2} L_{i_1 i_3 \dots i_k j}^{(k)} + \dots + \delta_{i i_k} L_{i_1 \dots i_{k-1} j}^{(k)} \right] \nabla_j v_i \end{aligned} \quad (18)$$


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The contribution from the last term in the left-hand side of Eq. (17) has a similar structure but contains the  $k-1$ th moment  $\mathbf{L}^{(k-1)}$  instead of  $\mathbf{L}^{(k)}$  in Eq. (18). Since the first moment of the distribution function  $f_{\mathbf{p}}^L$  equals to zero (see Eq.(10)) this contribution vanishes in equation for the second moment. Hence, in the case of  $k=2$  one has the following equation for the moments:

$$\begin{aligned} D_t L_{ij}^{(2)} + L_{ij}^{(2)} \nabla_k v_k + L_{ik}^{(2)} \nabla_k v_j + L_{kj}^{(2)} \nabla_k v_i \\ + \nabla_k L_{ijk}^{(3)} = I_{ij}^{(2)}, \end{aligned} \quad (19)$$

where  $I_{ij}^{(2)}$  is the second moment of the collision integral.

By the definition (11), all moments  $L_{i_1 \dots i_k}^{(k)}$  are sym-

In the derivation of Eqs. (13)-(14) the constraints due to conservation of particles, momentum and energy in a collision process

$$\sum_{\mathbf{p}} I_{\mathbf{p}} = 0, \quad \sum_{\mathbf{p}} p_i I_{\mathbf{p}} = 0, \quad \sum_{\mathbf{p}} \mathbf{p}^2 I_{\mathbf{p}} = 0, \quad (15)$$

have been used. The second moment  $L_{ij}^{(2)}$  is a stress tensor, which is related to the pressure  $P$  of an electron gas as  $Tr L_{ij}^{(2)} = 3P$ . This tensor can be decomposed into a scalar and a traceless parts

$$L_{ij}^{(2)} = P \delta_{ij} + \pi_{ij}, \quad Tr \pi_{ij} = 0. \quad (16)$$

The equation of motion (14) allows to eliminate the electrostatic potential  $\varphi$  from the kinetic equation (12)

$$D_t f_{\mathbf{p}}^L + \frac{p_i}{m} \nabla_i f_{\mathbf{p}}^L - \left( p_j \nabla_j v_i - \frac{\nabla_j L_{ij}^{(2)}}{n} \right) \frac{\partial f_{\mathbf{p}}^L}{\partial p_i} = I_{\mathbf{p}}^L \quad (17)$$

and, consequently, from all equations for higher moments. The term in brackets in Eq. (17) has a clear physical meaning. This is a force which acts on a particle in the Lagrange frame.

Calculation of the  $k$ th moment of the kinetic equation (17) leads to the equation for the  $k$ th moment of the distribution function  $L_{i_1 \dots i_k}^{(k)}$ . The  $k$ th moment of the first term in the left hand side of Eq. (17) gives the substantial derivative of the  $k$ th moment of the distribution function  $\mathbf{L}^{(k)}$ . Similarly the divergency of the  $k+1$ th moment  $\mathbf{L}^{(k+1)}$  comes from the second term. Contributions from the last two terms on the left have more complicated tensor structure. For example, the  $k$ th moment of the first term in the brackets in Eq. (17) is

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metric with respect to transmutation of all indexes. All equations for moments must preserve this symmetry. To explicitly incorporate this condition it is convenient to introduce special notations for two possible symmetric products of a symmetric  $k$ -th rank tensor  $T_{i_1 \dots i_k}^{(k)}$  and a vector  $a_j$ . The first one is the dot product, which decreases the rank of a tensor from  $k$  to  $k-1$  and corresponds to the contraction of the vector index  $j$  with one of tensor indexes

$$\mathbf{T}^{(k)} \cdot \mathbf{a} \equiv T_{i_1 \dots i_{k-1} j}^{(k)} a_j. \quad (20)$$

We also define a symmetric direct product of  $\mathbf{T}^{(k)}$  and  $\mathbf{a}$  which gives a tensor of the rank  $k+1$  and is equal to

the sum over all transmutations of the vector index  $j$  and tensor indexes  $i_1 \dots i_k$

$$\{\mathbf{T}^{(k)} \otimes \mathbf{a}\}_S \equiv T_{i_1 \dots i_k}^{(k)} a_j + T_{j i_2 \dots i_k}^{(k)} a_{i_1} + T_{i_1 j \dots i_k}^{(k)} a_{i_2} + \dots + T_{i_1 \dots i_{k-1} j}^{(k)} a_{i_k}. \quad (21)$$

These notations allow us to rewrite the moments of different terms of the kinetic equation (17) in a compact way. For example, the  $k$ th moment of the first term in the round brackets in Eq. (17), which is given by Eq. (18) above, can be expressed as

$$\frac{1}{m^{k-1}} \sum_{\mathbf{p}} p_{i_1} \dots p_{i_k} p_j \frac{\partial f_{\mathbf{p}}^L}{\partial p_i} \nabla_j v_i = \mathbf{L}^{(k)} (\nabla \cdot \mathbf{v}) + \{(\mathbf{L}^{(k)} \cdot \nabla) \otimes \mathbf{v}\}_S.$$

Similarly, the moment of the second term in the brackets in Eq. (17) is

$$\frac{1}{m^{k-1}} \sum_{\mathbf{p}} p_{i_1} \dots p_{i_k} \frac{\partial f_{\mathbf{p}}^L}{\partial p_i} \nabla_j L_{ij}^{(2)} = \frac{1}{m} \{\mathbf{L}^{(k-1)} \otimes (\nabla \cdot \mathbf{L}^{(2)})\}_S$$

Finally the system of equations for the moments takes the form

$$D_t n + n \nabla \cdot \mathbf{v} = 0, \quad (22)$$

$$m n D_t \mathbf{v} + \nabla \cdot \mathbf{L}^{(2)} - e n \nabla \varphi = 0, \quad (23)$$

$$D_t \mathbf{L}^{(2)} + \mathbf{L}^{(2)} (\nabla \cdot \mathbf{v}) + \{(\mathbf{L}^{(2)} \cdot \nabla) \otimes \mathbf{v}\}_S + \nabla \cdot \mathbf{L}^{(3)} = \mathbf{I}^{(2)} \quad (24)$$

$$D_t \mathbf{L}^{(k)} + \mathbf{L}^{(k)} (\nabla \cdot \mathbf{v}) + \{(\mathbf{L}^{(k)} \cdot \nabla) \otimes \mathbf{v}\}_S + \nabla \cdot \mathbf{L}^{(k+1)} - \frac{1}{m n} \{\mathbf{L}^{(k-1)} \otimes (\nabla \cdot \mathbf{L}^{(2)})\}_S = \mathbf{I}^{(k)}, \quad (25)$$

where the last equation (25) is valid for  $k > 2$ .

In a collision dominated case when conditions (6) are fulfilled, the high moments are small and the infinite chain of equations (22)-(25) can be truncated (Chapman-Enskog or Grad methods, see Ref. 14). As the traceless part  $\pi_{ij}$  of the stress tensor (16) and the third moment tensor  $L_{ijk}^{(3)}$  vanish for the locally equilibrium distribution function, they remain small under conditions (6). In this case tensors  $\pi_{ij}$  and  $L_{ijk}^{(3)}$  describe, respectively, viscosity and thermal conductivity in a collision dominated liquid. However, if conditions (6) are violated,  $\pi_{ij}$  and  $L_{ijk}^{(3)}$  as well as the higher-order moments are not small. Yet the infinite chain of equations can be decoupled if all physical quantities are slow varying functions of  $\mathbf{r}$ . It is seen from Eq. (19) that the third moment  $L_{ijk}^{(3)}$  enters only under a spatial derivative, hence its contribution is proportional to  $1/L$ . This remains true for all higher moments in higher-order equations (25). Thus the truncation of the chain can be guaranteed by the smallness of the gradients of the moments instead of the smallness of the moments itself. The dimensionless parameter of this expansion is

$$\gamma \sim \frac{u}{L \max\{\omega, \nu_c\}} \ll 1. \quad (26)$$

It is clear that both inequalities (6) can be violated while the condition (26) is fulfilled. For collisionless nondegenerate plasma the possibility to decouple the chain of equations for moments based on the condition similar to Eq. (26) was first pointed out in Ref. 16, where it was referred to as “the low temperature approximation”.

To justify the consistency of this truncation procedure we estimate the order of magnitude of the terms in Eq. (25). According to Eq. (22), the spatial derivative of the velocity field  $\mathbf{v}$  is of the order of the inverse characteristic time  $1/t_{pr} \sim \omega$  hence the first three terms in Eq. (25) are proportional to  $\omega$ . The right-hand side of this equation is evidently proportional to the collision frequency  $\nu_c$ . The last two terms in the left-hand side contain only spatial derivatives and are of the order of  $u/L$ . Thus the contribution of  $\mathbf{L}^{(k)}$  in the equation for  $\mathbf{L}^{(k-1)}$  contains an additional smallness  $\gamma$  (26), which means that the correction originating from  $\mathbf{L}^{(k)}$  to the equation for the density (22) is of the order  $\gamma^k$ . Therefore to obtain a theory valid up to  $\gamma^k$  one should keep  $k+1$  equations for  $\mathbf{L}^{(q)}$  ( $0 \leq q \leq k$ ) with the contribution of the  $k+1$ -th moment and the spatial derivative of  $\mathbf{L}^{(2)}$  being omitted in the last equation. The resulting theory is a generalization of hydrodynamics which is valid far from the equilibrium.

It should be mentioned that, in principle, the described truncation procedure may not provide the closed system of equations because the moments of the collision integral are, in general, the functionals of the distribution function. However, for a high frequency process ( $\omega/\nu_c \gg 1$ ) the contribution from the collision term disappears and we obtain the closed set of equations of the “collisionless hydrodynamics” which corresponds to the Vlasov limit of the kinetic equation. The collision terms become eventually important on the long time scale, when the system approaches the equilibrium. In this limit the moments of the collision integral are linear functions of the corresponding moments of the distribution function.<sup>14</sup> Hence under the condition ( $\omega/\nu_c \ll 1$ ) they can be presented in a close form. Matching these two limits gives the hydrodynamical theory which correctly describes the high- and the low-frequency limit and is approximately valid for all values of  $\omega/\nu_c$ .

There is an obvious relation of the generalized hydrodynamics to the standard linear response theory. In the high-frequency region the collisionless hydrodynamics is a regular expansion of Vlasov equation in terms of parameter (26). Consequently in the case of a weak perturbation the linearized hydrodynamics equations should give the same results as the linearized Vlasov equation. For example, the response function obtained from the linearized system (22)-(25) which is truncated at  $k$ th equation should coincide with the microscopic response function up to  $(u/L\omega)^k \sim (uq/\omega)^k$  (where  $q \sim 1/L$  is a wave vector of a perturbation). However the hydrodynamics in

its general form goes beyond the linear response theory (RPA) since it is valid for nonlinear regime.

In this section we introduced parameter  $\gamma$ , which governs the decoupling procedure. The smallness of this parameter restricts the region of applicability of the generalized hydrodynamics. The condition (26) physically means that a particle passes the characteristic length scale  $L$  in a time much longer than the typical time of the process  $\sim 1/\omega$  or a collisional time  $\sim 1/\nu_c$ . In this case there are no particles which move in resonance with a collective motion and may provide an energy exchange between single-particle and collective excitation. In other words, the contribution of the Landau damping to the evolution of the system is small. This is the physical restriction of the applicability of the generalized hydrodynamics which is (as any hydrodynamics) the theory of a collective motion.

### III. THE SECOND-ORDER THEORY. HYDRODYNAMICS OF A DEGENERATE FERMI GAS

The theory of the second order with respect to parameter  $\gamma$  (26) corresponds to neglect the third moment  $\mathbf{L}^{(3)}$  in Eq. (24). Let us decompose the stress tensor into the scalar and the traceless parts (16) and write the system of equations in terms of the fields  $P$  and  $\boldsymbol{\pi}$ :

$$D_t n + n \nabla \cdot \mathbf{v} = 0, \quad (27)$$

$$m D_t \mathbf{v} + \nabla P + \nabla \cdot \boldsymbol{\pi} - en \nabla \varphi = 0, \quad (28)$$

$$D_t P + \frac{5}{3} P \nabla \cdot \mathbf{v} + \frac{2}{3} (\boldsymbol{\pi} \cdot \nabla) \cdot \mathbf{v} = 0, \quad (29)$$

$$D_t \boldsymbol{\pi} + \boldsymbol{\pi} (\nabla \cdot \mathbf{v}) + \{(\boldsymbol{\pi} \cdot \nabla) \otimes \mathbf{v}\}_S - \mathbf{1} \frac{2}{3} (\boldsymbol{\pi} \cdot \nabla) \cdot \mathbf{v} + P \left( \{ \nabla \otimes \mathbf{v} \}_S - \mathbf{1} \frac{2}{3} \nabla \cdot \mathbf{v} \right) = \mathbf{I}^{(2)}, \quad (30)$$

where  $\mathbf{1}$  is the unit tensor. Eqs. (29) and (30) are the trace and the traceless part of Eq. (24) respectively. Due to the conservation of energy in a collision process the second moment of the collision integral  $\mathbf{I}^{(2)}$  is a traceless tensor. Consequently it contributes only to Eq. (30). Eqs. (27)-(30) are the hydrodynamics equations which contain two scalars  $n$  and  $P$ , one vector  $\mathbf{v}$  and one traceless tensor  $\boldsymbol{\pi}$ . For a charged liquid one has to add the Poisson equation (3) for the scalar potential  $\varphi$ .

Eigenmodes of a charge liquid (plasma waves) lie in a high frequency region where the collisionless limit is reached. In Sec. IV (see also Ref. 13) it is shown that the system of equation (27)-(30) leads to the correct plasmon dispersion up to the second order of parameter  $\gamma$ . This is a reflection of the fact that the linearized Eqs. (27)-(30) correspond to a correct linear response theory up to the  $\gamma^2$ .

In the low frequency limit an expression for collision integral is needed. To demonstrate how a long-time scale

behavior is recovered, we take  $I_{\mathbf{p}}$  in a Krook-Bhatnager-Gross (KBG) approximation (see, i.e. Ref. 17)

$$I_{\mathbf{p}}[f_{\mathbf{p}}] = -\nu_c (f_{\mathbf{p}}(\mathbf{r}, t) - f_{\mathbf{p}}^F(\mathbf{r}, t)), \quad (31)$$

where  $f_{\mathbf{p}}^F$  is a local equilibrium Fermi function with position-dependent velocity, chemical potential  $\mu(\mathbf{r}, t)$  and temperature  $T(\mathbf{r}, t)$ . By the definition the function  $f_{\mathbf{p}}^F$  is chosen to give the same values of velocity  $\mathbf{v}(\mathbf{r}, t)$ , density  $n(\mathbf{r}, t)$  and pressure  $P(\mathbf{r}, t)$ , as the exact distribution function  $f_{\mathbf{p}}(\mathbf{r}, t)$ :

$$n(\mathbf{r}, t) = \sum_{\mathbf{p}} f_{\mathbf{p}} = \sum_{\mathbf{p}} f_{\mathbf{p}}^F, \quad (32)$$

$$\mathbf{j}(\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{\mathbf{p}}{m} f_{\mathbf{p}} = \sum_{\mathbf{p}} \frac{\mathbf{p}}{m} f_{\mathbf{p}}^F, \quad (33)$$

$$P(\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{3m} f_{\mathbf{p}+m\mathbf{v}} = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{3m} f_{\mathbf{p}+m\mathbf{v}}^F. \quad (34)$$

Eqs. (32-34) guarantee that the collision integral (31) satisfies the general properties (15).

The second moment of  $I_{\mathbf{p}}$  (31) is equal to

$$\mathbf{I}^{(2)} = -\nu_c \left( \mathbf{L}^{(2)} - \mathbf{1} P_F(n, T) \right), \quad (35)$$

where  $P_F(n, T)$  is the pressure of a Fermi gas with the distribution function  $f_{\mathbf{p}}^F$ . By definition (34), this pressure equals to the exact pressure  $P_F(n, T) = P \equiv \frac{1}{3} T r \mathbf{L}^{(2)}$ , hence the expression in the brackets in (35) equals to the traceless tensor  $\boldsymbol{\pi}$ . Consequently, Eq. (35) for  $\mathbf{I}^{(2)}$  is equivalent to two equations

$$\mathbf{I}^{(2)} = -\nu_c \boldsymbol{\pi}, \quad (36)$$

$$P = P_F(n, T). \quad (37)$$

Since  $P_F(n, T)$  is an equilibrium pressure for given  $n$  and  $T$ , the equation (37) may seem to be related to the assumption of the local equilibrium. In fact, it is not. Eq. (37) simply introduces the new independent scalar variable  $T(\mathbf{r}, t)$  instead of  $P(\mathbf{r}, t)$ . The system can still be in an arbitrary (up to  $\gamma^2$ ) nonequilibrium state. In the second order theory such state is uniquely described by two scalar, one vector and one traceless tensor functions. The traceless second rank tensor  $\boldsymbol{\pi}$  is responsible for deviations from the local equilibrium. In a degenerate Fermi system this tensor describes deviations of the shape of the Fermi surface from sphere.

The system of equations (27)-(30), with the collision term (36) and  $P$  from Eq. (37) transforms into the common hydrodynamics theory in the limit  $\omega/\nu_c \ll 1$ . Indeed, as we already mentioned above the spatial derivative of  $\mathbf{v}$  has the order of magnitude of  $\omega$ . Hence, the first four terms in the left hand side of (30) are proportional to  $\omega \boldsymbol{\pi}$ , the last term in the left hand side is proportional to  $\omega P$  and the right hand side is equal to  $\nu_c \boldsymbol{\pi}$ . Consequently, in a zero order of  $\omega/\nu_c$  Eq. (30)

leads to  $\pi_{ij} = 0$ , and Eqs.(27)-(29) become identical to hydrodynamics equations for an ideal liquid:

$$D_t n + n \nabla \mathbf{v} = 0, \quad (38)$$

$$mn D_t \mathbf{v} + \nabla P - en \nabla \varphi = 0, \quad (39)$$

$$D_t P + \frac{5}{3} P \nabla \cdot \mathbf{v} = 0. \quad (40)$$

Here Eq. (39) is the Euler equation and Eq. (40) is the equation for the conservation of energy. For a degenerate Fermi gas this set of equations exactly corresponds to the BHT. We emphasize again, that it is valid only in an extremely collision dominated regime.

In the first order of  $\omega/\nu_c \ll 1$  one has to neglect the first four terms in the left hand side of Eq. (30) in comparison with the last term which is proportional to the pressure  $P$ . Thus, the first order solution of Eq. (30) takes the form of the viscosity tensor

$$\pi_{ij} = -\frac{P}{\nu_c} \left( \nabla_i v_j + \nabla_j v_i - \frac{2}{3} \delta_{ij} \nabla_k v_k \right), \quad (41)$$

with the viscosity coefficient  $\eta = P/\nu_c$ . In this case Eqs. (28) and (29) are equivalent to Navier-Stokes equation and the equation for energy conservation in a viscous liquid respectively. In Eq. (41) we have turned back to common tensor notations to make the relation to the ordinary viscous hydrodynamics more transparent.

The formulas (36), (37) were obtained using the KBG collision integral (35). They are, however, more general than the KBG approximation itself. As it has been mentioned at the end of the previous section, the reason is that in the low frequency range, where collisions are important, the second moment  $I_{ij}^{(2)}$  is always a linear function of  $\pi_{ij}$ .<sup>14</sup> The coefficient  $\nu_c$  can thus be considered as a phenomenological parameter, which is related to viscosity  $\eta$  as  $\nu_c = P/\eta$ .

Eqs. (27)-(30) with  $\mathbf{I}^{(2)}$  from (36) and  $P$  from (37) constitute the closed set of equations of generalized hydrodynamics in the second order of  $\gamma$ . This set of equations gives a correct description of the high-frequency collisionless regime. It also leads to the classical hydrodynamics of a viscous fluid in a low-frequency limit. Consequently it should be valid, with a reasonable accuracy, for intermediate regime.

However, the second order theory has an inherent inconsistency. Though in a low frequency limit the viscous term is recovered correctly, the thermal conductivity contribution  $\nabla^2 T$  (Ref. 9) is still missing in the energy conservation equation. Without this contribution the correct static limit  $\nabla T(\mathbf{r}) = 0$  cannot be recovered. In fact, the term  $\nabla^2 T$  corresponds to correction of the fourth order of  $\gamma$  in the continuity equation. Physically this means an absence of a dissipative flow of energy with an accuracy up to  $\gamma^2$ . To include the thermal conductivity one has to consider the third and the fourth moments and neglect the fifth moment (see Sec. VI below).

The theory of the second order in  $\gamma$  is obviously applicable for a description of dynamics of a dense degenerate Fermi gas. In this case the condition  $T/E_F \ll 1$  ( $E_F$  is the local Fermi energy) is always satisfied and it is unnecessary to take into account the third and the fourth moments. At  $T = 0$  Eq. (37) reduces to  $P = \frac{1}{5m}(3\pi^2)^{2/3} n^{5/3}$  and Eq. (30) transforms into

$$(\boldsymbol{\pi} \cdot \nabla) \cdot \mathbf{v} = 0. \quad (42)$$

The equation for pressure  $P$  and the condition (42) together with the system of differential equations

$$D_t n + n \nabla \cdot \mathbf{v} = 0, \quad (43)$$

$$m D_t \mathbf{v} + \nabla P + \nabla \cdot \boldsymbol{\pi} - en \nabla \varphi = 0, \quad (44)$$

$$D_t \boldsymbol{\pi} + \boldsymbol{\pi}(\nabla \cdot \mathbf{v}) + \{(\boldsymbol{\pi} \cdot \nabla) \otimes \mathbf{v}\}_S - \mathbf{1} \frac{2}{3} (\boldsymbol{\pi} \cdot \nabla) \cdot \mathbf{v} + P \left( \{\nabla \otimes \mathbf{v}\}_S - \mathbf{1} \frac{2}{3} \nabla \cdot \mathbf{v} \right) = -\nu_c \boldsymbol{\pi}, \quad (45)$$

provide a complete system of equations of the generalized hydrodynamics for a degenerate Fermi gas.

Most applications of the Bloch's hydrodynamics concerned just the case of the degenerate Fermi gas. The set of equations (42)-(45) of generalized hydrodynamics should replace the BHT, which is incorrect except the collision dominated limit or the case of spatially homogeneous system.

There is an interesting relation of the linearized version of the generalized hydrodynamics in a high-frequency (collisionless) limit to the theory of elasticity. In the limit  $\omega/\nu_c \gg 1$  linearization of the system (27)-(30) gives

$$\partial_t \delta n + n_0 \nabla_k v_k = 0 \quad (46)$$

$$mn_0 \partial_t v_i + \nabla_i \delta P + \nabla_j \pi_{ij} - en_0 \nabla_i \varphi = 0 \quad (47)$$

$$\partial_t \delta P + \frac{5}{3} P_0 \nabla_k v_k = 0 \quad (48)$$

$$\partial_t \pi_{ij} + P_0 \left( \nabla_i v_j + \nabla_j v_i - \frac{2}{3} \delta_{ij} \nabla_k v_k \right) = 0, \quad (49)$$

where  $\delta n$  and  $\delta P$  are deviations from the equilibrium density  $n_0$  and pressure  $P_0$ . Let us introduce a displacement vector  $\mathbf{u}(\mathbf{r}, t)$  as  $\delta n = -n_0 \nabla \cdot \mathbf{u}$ . The continuity equation (46) gives a usual relation between velocity and displacement  $\partial_t \mathbf{u} = \mathbf{v}$ . Introducing a stress tensor  $\sigma_{ij} = -\delta P \delta_{ij} - \pi_{ij}$  one can rewrite Eq. (47) as

$$mn_0 \partial_t^2 u_i - \nabla_j \sigma_{ij} - en_0 \nabla_i \varphi = 0. \quad (50)$$

The relationship of the tensor  $\sigma_{ij}$  to the displacement  $u_i$  follows from Eqs. (48, 49) and takes the same form as in the elasticity theory<sup>18</sup>

$$\sigma_{ij} = K \nabla_k u_k \delta_{ij} + \mu \left( \nabla_i u_j + \nabla_j u_i - \frac{2}{3} \delta_{ij} \nabla_k u_k \right), \quad (51)$$

where the bulk modulus  $K$  and the shear modulus  $\mu$  of an electron gas are  $K = \frac{5}{3} P_0$  and  $\mu = P_0$ . Physically the

bulk modulus is responsible for the increase of the energy which is caused by the local change of the occupied volume in a momentum space, whereas the shear stress describes the deviation of the shape of this volume from a sphere. In the next section we derive the hydrodynamics for a Fermi liquid and find the correlation contribution to bulk and shear modulus. The elastic description of Fermi systems has been recently discussed in Ref. 15 within the linear response theory.

Inclusion of collisions violates the exact correspondence to the elasticity theory. In this case the shear stress tensor  $\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} = -\pi_{ij}$  should be determined from the equation

$$\partial_t \pi_{ij} + \mu \partial_t \left( \nabla_i u_j + \nabla_j u_i - \frac{2}{3} \delta_{ij} \nabla_k u_k \right) = -\nu_c \pi_{ij}, \quad (52)$$

which follows from Eqs. (49) and (36). Eq. (52) exactly coincides with the equation for the shear stress tensor in the phenomenological theory of *highly viscous fluids* by Maxwell.<sup>18</sup> These fluids (for instance, glycerin or resin) behave as solids at short intervals of time, but as viscous liquids on a large time scale (see Ref. 18). The generalized hydrodynamics thus provide a surprising but clear analogy between an electron gas and Maxwellian highly viscous fluids.

#### IV. THE GENERALIZED HYDRODYNAMICS OF A FERMIL LIQUID

In this section we develop the hydrodynamic description of a Fermi liquid based on the Landau theory. Since the Landau theory is valid only for small deviations of the quasiparticle distribution function  $n_{\mathbf{p}}$  from the Fermi function, only linearized version of the generalized hydrodynamics can be constructed. Although such theory is formally equivalent to the linear response approach, the hydrodynamical formulation is generally more efficient for spatially inhomogeneous problems. In addition, the derivation of the generalized hydrodynamics of a Fermi liquid is of a general interest since it transparently shows how correlation effects contribute to the stress tensor.

We start from a kinetic equation for a charged Fermi liquid<sup>19</sup>

$$\partial_t n_{\mathbf{p}} + \frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{p}} \nabla n_{\mathbf{p}} - \nabla \varepsilon_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} + e \nabla \varphi \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} = I_{\mathbf{p}}, \quad (53)$$

where  $\varepsilon_{\mathbf{p}}$  is the local quasiparticle energy, which depends on the distribution function and, consequently, is a function of spatial coordinates. The gradient of this function gives an additional force which has its origin in the correlation effects.

Following the derivation of the hydrodynamic equations for a Fermi gas (Sec. II) we separate the macroscopic and relative motion of a liquid by transformation

to the comoving frame. The kinetic equation for the distribution function  $n_{\mathbf{p}}^L$  in this frame looks similar to Eq. (12), but with an additional correlation force:

$$D_t n_{\mathbf{p}}^L + \frac{\partial \varepsilon_{\mathbf{p}}^L}{\partial \mathbf{p}} \nabla n_{\mathbf{p}}^L - (\mathbf{p} \nabla) \mathbf{v} \frac{\partial n_{\mathbf{p}}^L}{\partial \mathbf{p}} - (m D_t \mathbf{v} - e \nabla \varphi + \nabla \varepsilon_{\mathbf{p}}^L) \frac{\partial n_{\mathbf{p}}^L}{\partial \mathbf{p}} = I_{\mathbf{p}}^L \quad (54)$$

According to the Landau assumption, the quasiparticle energy is a linear functional of the deviation of the distribution function from the Fermi step function:

$$\varepsilon_{\mathbf{p}}^L = \varepsilon_{\mathbf{p}}^{(0)} + \sum_{\mathbf{p}'} f_{\mathbf{p}'\mathbf{p}} \delta n_{\mathbf{p}'}^L, \quad (55)$$

$$n_{\mathbf{p}}^L = n_{\mathbf{p}}^{(0)} + \delta n_{\mathbf{p}}^L, \quad n_{\mathbf{p}}^{(0)} = \theta(p - p_F), \quad (56)$$

where  $\varepsilon_{\mathbf{p}}^{(0)}$  is the quasiparticle energy in the undisturbed system and  $f_{\mathbf{p}'\mathbf{p}}$  is the quasiparticle-quasiparticle interaction. Linearization of the kinetic equation (54) leads to the Landau-Silin equation,<sup>19</sup> written in a comoving frame

$$\partial_t \delta n_{\mathbf{p}}^L + \frac{p_i}{m^*} \nabla_i \delta n_{\mathbf{p}}^L - p_j \nabla_j v_i \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial p_i} - \left( m \partial_t v_i - e \nabla_i \varphi + \sum_{\mathbf{p}'} f_{\mathbf{p}'\mathbf{p}} \nabla_i \delta n_{\mathbf{p}'}^L \right) \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial p_i} = I_{\mathbf{p}}^L. \quad (57)$$

The quasiparticle mass  $m^*$ , which enters equation (57), is related to the bare fermion mass  $m$  as follows<sup>19</sup>

$$\left( \frac{\partial^2}{\partial p^2} \varepsilon_{\mathbf{p}}^{(0)} \right)^{-1} \equiv m^* = m \left( 1 + \frac{1}{3} F_1 \right). \quad (58)$$

where  $F_1$  is the first of Landau Fermi-liquid parameters:

$$F_l = \frac{m^* p_F}{\pi^2} \int \frac{d\Omega'}{4\pi} f_{\mathbf{p}'\mathbf{p}} P_l(\cos \theta), \quad (59)$$

where  $\theta$  is the angle between  $\mathbf{p}$  and  $\mathbf{p}'$ ,  $d\Omega'$  is an element of a solid angle around  $\mathbf{p}'$  and  $P_l(\cos \theta)$  are Legendre polynomials.

Zeroth and first moments of Eq. (57) correspond to the continuity equation and the equation for velocity of a Fermi liquid:

$$\partial_t \delta n + n_0 \nabla_i v_i = 0 \quad (60)$$

$$m n_0 \partial_t v_i + \nabla_j \delta P_{ij} - e n_0 \nabla_i \varphi = 0 \quad (61)$$

In Eq. (61) we introduced the stress tensor of a Fermi liquid

$$\delta P_{ij} = \sum_{\mathbf{p}} \frac{p_i p_j}{m^*} \delta n_{\mathbf{p}}^L - \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}'\mathbf{p}} \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial p_i} p_j \delta n_{\mathbf{p}'}^L \quad (62)$$

The first term in Eq. (62) corresponds to the kinetic stress tensor of quasiparticles, whereas the second term

describes the interaction of quasiparticles and is responsible for a correlation contribution to the stress.

According to the results of the previous section, the second order approximation (up to  $\gamma^2$ ) gives a consistent hydrodynamical theory of a degenerate Fermi system. To make the system of equations (60)-(61) complete one needs only an equation for the second moment or the stress tensor. To simplify the derivation of this equation we separate the stress tensor  $\delta P_{ij}$  into the scalar  $\delta P$  and the traceless  $\pi_{ij}$  parts

$$\delta P_{ij} = \delta P \delta_{ij} + \pi_{ij} \quad (63)$$

First we rewrite (62) as follows

$$\delta P_{ij} = \frac{1}{m^*} \sum_{\mathbf{p}} T_{ij}(\mathbf{p}) \delta n_{\mathbf{p}}^L \quad (64)$$

$$T_{ij}(\mathbf{p}) = p_i p_j + \frac{m^* p_F}{\pi^2} \int \frac{d\Omega'}{4\pi} f_{\mathbf{p}'\mathbf{p}} p'_i p'_j \quad (65)$$

In the Landau theory all momentums reside on the Fermi surface. By symmetry tensor  $T_{ij}(\mathbf{p})$  is uniquely representable in the form

$$T_{ij}(\mathbf{p}) = \alpha p^2 \delta_{ij} + \beta (p_i p_j - \frac{1}{3} p^2 \delta_{ij}), \quad (66)$$

where the constants  $\alpha$  and  $\beta$  are determined by the following two equations

$$\frac{1}{p_F^2} T_{ii} = 3\alpha = 1 + \frac{m^* p_F}{\pi^2} \int \frac{d\Omega'}{4\pi} f_{\mathbf{p}'\mathbf{p}}, \quad (67)$$

$$\begin{aligned} \frac{1}{p_F^4} p_i T_{ij} p_j &= \alpha + \frac{2}{3} \beta = \\ &1 + \frac{m^* p_F}{\pi^2} \int \frac{d\Omega'}{4\pi} f_{\mathbf{p}'\mathbf{p}} \frac{(\mathbf{p}\mathbf{p}')^2}{p_F^4}. \end{aligned} \quad (68)$$

Solutions to these equations are

$$\alpha = 1 + F_0, \quad \beta = 1 + \frac{1}{5} F_2 \quad (69)$$

Substituting  $\alpha$  and  $\beta$  (69) into Eqs. (66),(64) we obtain the following microscopic expressions for  $\delta P$  and  $\pi_{ij}$

$$\delta P = (1 + F_0) \sum_{\mathbf{p}} \frac{p^2}{3m^*} \delta n_{\mathbf{p}}^L \equiv v_s^2 \delta \rho, \quad (70)$$

$$\pi_{ij} = (1 + \frac{1}{5} F_2) \sum_{\mathbf{p}} \left\{ \frac{p_i p_j}{m^*} - \frac{p^2}{3m^*} \delta_{ij} \right\} \delta n_{\mathbf{p}}^L, \quad (71)$$

where  $v_s^2 = (1 + F_0) p_F^2 / 3mm^*$  is the square of velocity of sound<sup>19</sup> and  $\delta \rho = m \delta n$  is a variation of the mass density. The equivalence in Eq. (70) shows that the scalar part of the stress tensor  $\delta P$  exactly equals to the variation of the pressure of the Fermi liquid.

Expressions for  $\delta P$  and  $\pi_{ij}$  (70) and (71) have the same form as for the gas case except the factors in front of the sums and the change of the bare mass to the mass of

a quasiparticle. Thus the derivation of an equation for the stress tensor is straightforward. The final set of the generalized hydrodynamics equations of a Fermi liquid in the second order in parameter  $\gamma$  (26) takes the form

$$\partial_t \delta n + n_0 \nabla_k v_k = 0 \quad (72)$$

$$mn_0 \partial_t v_i + \nabla_i \delta P + \nabla_j \pi_{ij} - en_0 \nabla_i \varphi = 0 \quad (73)$$

$$\partial_t \delta P + K \nabla_k v_k = 0 \quad (74)$$

$$\partial_t \pi_{ij} + \mu \left( \nabla_i v_j + \nabla_j v_i - \frac{2}{3} \delta_{ij} \nabla_k v_k \right) = -\nu_c \pi_{ij}. \quad (75)$$

Hence, in a Fermi liquid we have again the system of equations which is similar to the Maxwell's theory of highly viscous fluids, but with the bulk  $K$  and the shear  $\mu$  moduli directly related to microscopic Landau parameters

$$K = mn_0 v_s^2 = \frac{5}{3} P_0 \frac{1 + F_0}{1 + \frac{1}{3} F_1}, \quad (76)$$

$$\mu = P_0 \frac{1 + \frac{1}{5} F_2}{1 + \frac{1}{3} F_1}, \quad (77)$$

where  $P_0$  is a pressure of an ideal Fermi gas.

The bulk modulus (76) and the shear modulus (77) exactly coincide with those of Ref. 15, where they have been obtained by analyzing the linear response function. In ideal Fermi gas ( $F_l = 0$ ) Eqs. (76) and (77) transform to the ideal gas moduli (see Sec. III).

## V. HYDRODYNAMIC THEORY OF COLLECTIVE MODES

### A. Plasma waves in a homogeneous system

The frequencies of collective plasma modes in a charged liquid have an order of magnitude of  $\omega_p$  which commonly resides in the high frequency (collisionless) region  $\omega/\nu_c \gg 1$ . The plasma oscillations are solutions of the linearized system of the hydrodynamics equations. To obtain the dispersion of plasmons we have to solve Eqs. (72)-(75) with the collision term being omitted in Eq. (75). In the collisionless limit it is convenient to introduce a displacement vector  $\mathbf{u}$  and rewrite this system in the form of the elasticity theory (50), (51), where  $K$  and  $\mu$  are determined, in general, by Eqs. (76) and (77). Substituting Eq. (51) to Eq.(50) we have an equation for the displacement vector of a "charged elastic medium"

$$mn_0 \partial_t^2 \mathbf{u} - (K + \frac{1}{3} \mu) \nabla (\nabla \mathbf{u}) - \mu \nabla^2 \mathbf{u} - en_0 \nabla \varphi = 0, \quad (78)$$

which should be solved together with the Poisson equation

$$\nabla^2 \varphi = -4\pi e \nabla_0 \mathbf{u} \quad (79)$$

Considering a plane-wave solution  $e^{-i(\omega t - \mathbf{q}\mathbf{r})}$  we get the dispersion of the plasma waves

$$\omega^2(q) = \omega_p^2 + v_0^2 q^2 \quad (80)$$

where

$$v_0^2 = \frac{1}{mn_0} \left( K + \frac{4}{3} \mu \right) = v_s^2 + \frac{4\mu}{3mn_0}. \quad (81)$$

The first contribution in  $v_0^2$  equals to the square of the sound velocity  $v_s^2$  (the result of the ordinary hydrodynamics) and comes from the fluctuations of pressure  $\delta P$  (see formulas (70) and (76)). The second contribution arises from the traceless part of the stress tensor  $\pi_{ij}$ . The sum gives a correct coefficient which can be obtained from the longitudinal response function of a charged Fermi liquid.<sup>19</sup> In the case of a noninteracting gas at arbitrary degeneracy the expression for  $v_0^2$  reduces to the formula

$$v_0^2 = \frac{5P_0}{3mn_0} + \frac{4P_0}{3mn_0} = \frac{3P_0}{mn_0} \equiv < v_p^2 > .$$

which coincides with the result of the RPA (up to the second order of  $qu/\omega \sim \gamma$ ) and in a degenerate Fermi gas gives the well known result  $\frac{3}{5}v_F^2$ . It is straightforward to show that taking into account the third and fourth moments in the linearized system of equations leads to the correct plasmon dispersion up to  $q^4$ .

## B. Plasma oscillations of a confined electron liquid

In the case of one dimensional motion (e.g. for plasma waves in an infinite medium which are considered in the previous subsection) the displacement vector  $\mathbf{u}$  depends only on one coordinate. Hence, the equation of motion takes the following simple form:

$$\partial_t^2 \mathbf{u} - v_0^2 \nabla^2 \mathbf{u} - \frac{e}{m} \nabla \varphi = 0. \quad (82)$$

Eq. (82) coincides with that of the linearized Bloch's theory. The only difference is that the dispersion coefficient equals to the correct value  $v_0^2$  instead of a square of a velocity of sound  $v_s^2$ . It seems easy to phenomenologically improve the BHT. One could try to replace the dispersion coefficient by the correct value  $v_0^2$  and hence obtain a correct dispersion of plasmon.<sup>3,5,7,11</sup> However, from the general point of view such an improvement cannot be consistent since the correct static limit requires  $v_s^2$  as a coefficient in Eq. (82). The frequency-dependent coefficient was commonly used in hydrodynamics calculations to recover both the high- and the low-frequency regimes (see Ref. 11 and references therein). It has been then recognized<sup>11</sup> that the theory with a frequency dependent dispersion coefficient violates the harmonic potential theorem (HPT) in a spatially inhomogeneous case. Nonetheless, it was believed that since the theory with

the replaced coefficient gives the correct plasmon dispersion in the infinite medium it should hopefully give the proper description of the high-frequency plasma oscillations for any geometries of a confined electron gas.<sup>3,5,7</sup>

As we show in the previous sections, the consistent theory which is valid both in a high- and low-frequency regimes must inevitably include the tensor fields of higher rank (e.g.  $\pi_{ij}$  in the second order theory). At low frequency  $\pi_{ij}$  goes to zero that gives a correct static limit. However, in the high-frequency regime it does not contain any small parameter and gives the contribution of the same order of magnitude as the variation of the scalar pressure. Within the linearized equations the contribution of the tensor  $\pi_{ij}$  can be interpreted as a shear stress of a Fermi system. Formally this leads to more complicated structure of the differential operator in Eq. (78) than one of the operator expected from the BHT Eq. (82).

In the present subsection we consider two simple examples of a collective motion of a Fermi liquid which is confined in three and one dimensions. These examples show that the collective eigenfrequencies strongly differ from predictions of the standard hydrodynamics.

Consider a homogeneous (in equilibrium state) electron liquid which is confined by an external potential. The homogeneity of the equilibrium state can be reached for example with the help of a positively charged background of the density equal to the density of electrons  $n_0$  or with the parabolic potential of a proper frequency  $\omega_0$ . For example, in a three-dimensional spherically symmetric case (parabolic quantum dot) the frequency  $\omega_0 = \omega_p/\sqrt{3}$ . The two-dimensional (parabolic quantum wire) and one-dimensional (quantum well) potentials have frequencies  $\omega_0 = \omega_p/\sqrt{2}$  and  $\omega_0 = \omega_p$  respectively.

Eigenmodes are solutions to equations (50),(51),(79) with the proper boundary conditions. Integration of Eqs. (50) and (79) across the surface gives the set of boundary conditions

$$n_j \sigma_{ij}(\mathbf{r}_s) = 0, \quad (83)$$

$$\mathbf{n} \nabla \varphi^>(\mathbf{r}_s) - \mathbf{n} \nabla \varphi^<(\mathbf{r}_s) = 4\pi e n_0 \mathbf{n} \mathbf{u}(\mathbf{r}_s), \quad (84)$$

$$\varphi^>(\mathbf{r}_s) = \varphi^<(\mathbf{r}_s), \quad (85)$$

where  $\mathbf{r}_s$  is a coordinate of a surface point,  $\varphi^>$  and  $\varphi^<$  are respectively the electrostatic potential  $\varphi$  outside and inside the surface and  $\mathbf{n}$  is the unit vector normal to the surface. The boundary condition (83) corresponds to the "free surface" and allows an electron liquid to cross the boundary and oscillate with respect to the equilibrium position. For example, the rigid oscillations of an electron liquid  $\mathbf{u} = \mathbf{u}_0 e^{-i\omega t}$  ( $\mathbf{u}_0$  is a constant vector) satisfy Eqs. (50),(51),(79) and conditions (83)-(85). Such oscillations have the frequency  $\omega = \omega_0$ . This fact apparently reflects the HPT.

The above system of equations and boundary conditions also satisfy the HPT (or generalized Kohn theorem) in the most general formulation, which was suggested by

Dobson.<sup>11</sup> We demonstrate this for a Fermi liquid confined by a spherically symmetric harmonic potential with  $\omega_0 = \omega_p/\sqrt{3}$  and subjected to external homogeneous electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields. The HPT motion corresponds to rigid displacement  $\mathbf{u}(\mathbf{r}, \mathbf{t}) = \mathbf{u}_0(t)$  of the liquid. Vector  $\mathbf{u}_0(t)$  is obviously the center-of-mass coordinate. Since the stress tensor is equal to zero, the boundary condition (83) is trivially satisfied and equation of motion (50) has the form

$$m\partial_t^2 \mathbf{u}_0 - e\mathbf{H} \times \partial_t \mathbf{u}_0 + e\mathbf{E} - e\nabla\varphi^< = 0. \quad (86)$$

The scalar potential  $\varphi$  satisfies the Laplace equation and electrostatic boundary conditions (84), (85). For rigid motion these conditions coincide with the conditions on the boundary of a homogeneously polarized sphere with a polarization vector  $\mathbf{P} = -en_0\mathbf{u}_0$ . The soliton of this problem inside the sphere is a depolarization field

$$\nabla\varphi^< = \frac{4\pi}{3}\mathbf{P} = -\frac{4\pi en_0}{3}\mathbf{u}_0.$$

Substituting this solution into the Eq.(86) we find that the center-of-mass moves according to equation

$$m\partial_t^2 \mathbf{u}_0 - e\mathbf{H} \times \partial_t \mathbf{u}_0 + e\mathbf{E} + \omega_0^2 \mathbf{u}_0 = 0, \quad (87)$$

in exact correspondence with the HPT.<sup>11</sup>

To avoid a confusion we would like to outline that the boundary condition (83) assumes the parabolic potential outside the region occupied by an electron liquid. Physically it can be realized, for example, at the boundary of a depletion region in p-n junctions or in artificial quantum structures. The condition (83) is not directly applicable to a model surface of metals with abrupt change of the positive background at the boundary. The reason is a jump of a potential at the surface which forbids electrons to penetrate the boundary.

Let us separate longitudinal and transverse components of the displacement  $\mathbf{u}$

$$\mathbf{u} = \nabla\psi + \mathbf{u}_t, \quad \nabla\mathbf{u}_t = 0 \quad (88)$$

( $\psi$  is a potential of the longitudinal component) and rewrite the system of equations (50), (51), (79) and boundary conditions (83)-(85) in terms of variables  $\psi$  and  $\mathbf{u}_t$ . We are looking for harmonic in time solutions  $\sim e^{-i\omega t}$ .

In the region occupied by an electron liquid we have the system of differential equations:

$$\omega^2\psi + c_l^2\nabla^2\psi + \frac{e}{m}\varphi = 0, \quad (89)$$

$$\omega^2\mathbf{u}_t + c_t^2\nabla^2\mathbf{u}_t = 0, \quad \nabla\mathbf{u}_t = 0, \quad (90)$$

$$\nabla^2(\varphi + 4\pi en_0\psi) = 0. \quad (91)$$

Outside the surface the Laplace equation for the scalar potential  $\varphi$  must be satisfied

$$\nabla^2\varphi = 0. \quad (92)$$

All solutions of Eqs. (89)-(92) must fulfill the boundary conditions

$$(c_l^2 - 2c_t^2)n_i\nabla^2\psi + 2c_t^2n_j\nabla_j\nabla_i\psi + 2c_t^2n_j(\nabla_i u_t^j + \nabla_j u_t^i) = 0, \quad (93)$$

$$\mathbf{n}\nabla(\varphi^< + 4\pi en_0\psi) = \mathbf{n}\nabla\varphi^>, \quad (94)$$

$$\varphi^< = \varphi^>, \quad (95)$$

where all functions are taken at the surface. In Eqs. (89)-(95) we introduced the notations:

$$c_l^2 = v_0^2 = \frac{1}{mn_0}(K + \frac{4}{3}\mu), \quad (96)$$

$$c_t^2 = \frac{\mu}{mn_0}. \quad (97)$$

The boundary condition (93) demonstrates a peculiarity of the generalized hydrodynamics. Longitudinal and transverse oscillations are, in general, mixed at the surface. This effect is well known in the elasticity theory.<sup>9</sup> If the shear modulus is nonzero, the incident purely longitudinal (transverse) wave transforms into the mixed wave containing both longitudinal and transverse components. As a result the structure of eigenmodes is changed in comparison with the common hydrodynamics ( $\mu = 0$ ). The qualitative structure of the wave is not changed only in the case of a normal incidence (for example radial modes in the spherical symmetric system which are purely longitudinal). However, even in this case the shear modulus contributes to the boundary condition. The boundary condition for a purely longitudinal motion takes the form

$$(c_l^2 - 2c_t^2)n_i\nabla^2\psi + 2c_t^2n_j\nabla_j\nabla_i\psi = 0$$

which strongly differs from the common hydrodynamical condition  $\nabla^2\psi = 0$  for a parabolic potential.<sup>7</sup>

Bellow we solve equations (89)-(95) for two specific cases and show that nonzero shear modulus strongly influences the dispersion of the eigenmodes even for purely longitudinal waves.

### 1. Radial plasma oscillations in a parabolic quantum dot.

In a spherical symmetric quantum dot the normal vector  $\mathbf{n} = \hat{\mathbf{r}}$ . Thus all normal derivatives transform to the derivative with respect to the radial coordinate  $r$ . The direction of the displacement  $\mathbf{u}$  for radial modes coincides with  $\hat{\mathbf{r}}$ . Hence the transverse component  $\mathbf{u}_t$  equals to zero and functions  $\psi$  and  $\varphi$  depend only on  $r$ .

Radial solutions of Eqs. (89), (91), (92) which are regular at zero and at the infinity take the following general form

$$\psi(r) = A_1 \frac{\sin qr}{r} + B_1, \quad \varphi^<(r) = A_2 \frac{\sin qr}{r} + B_2, \quad (98a)$$

$$\varphi^>(r) = \frac{C}{r}, \quad (98b)$$

Substituting (98) into Eqs. (89), (91) and into boundary conditions for the scalar potential (94), (95) one gets the result

$$\psi(r) = A_1 \left\{ \frac{\sin qr}{r} - \frac{\omega_p^2}{\omega_p^2 + c_l^2 q^2} \frac{\sin qR}{R} \right\} \quad (99)$$

$$\varphi_<(r) = A_1 4\pi e n_0 \left\{ \frac{\sin qr}{r} - \frac{\sin qR}{R} \right\} \quad (100)$$

Here  $R$  is the radius of the quantum dot. The wave vector  $q$  is related to the frequency  $\omega$

$$\omega^2 = \omega_p^2 + c_l^2 q^2. \quad (101)$$

The last boundary condition (93) in the spherical symmetric case reads

$$(c_l^2 - 2c_t^2) \nabla^2 \psi(R) + 2c_t^2 \partial_r^2 \psi(R) = 0.$$

This equation together with the solution (99) gives the final dispersion equation

$$\tan qR = \frac{qR}{1 - (c_l/2c_t)^2 q^2 R^2} \quad (102)$$

which determine the allowed values of  $q$  and, consequently, the frequencies of the eigenmodes (101). The corresponding result of BHT can be obtained from Eq. (102) in the limit  $c_t \rightarrow 0$  and reads

$$\tan qR = 0.$$

The quantity  $(c_l/2c_t)^2$  which governs the difference of the last two dispersion equations can be expressed in terms of Landau parameters:

$$(c_l/2c_t)^2 = \frac{1}{3} \left( 1 + \frac{5}{4} \frac{1 + F_0}{1 + F_2/5} \right).$$

This value is obviously far from infinity (as assumed in the BHT). For example, in the case of a Fermi gas ( $F_l = 0$ ) we have  $(c_l/2c_t)^2 = 3/4$  which is less than one.

## 2. Surface plasma modes at the edge of a parabolic potential well.

Let us consider the surface plasma oscillations at the edge of an electron system confined by a one dimensional parabolic potential with  $\omega_0 = \omega_p$ . The situation is common in p-n junctions where the potential in the depletion region is parabolic and density of electrons (holes) in n (p) regions is approximately constant up to the corresponding boundary of the depletion region. Possible applications of the results to the surface modes of the parabolic potential well is obvious. Within BHT this problem has been considered in Ref. 7.

We suppose that the electron gas occupies the lower half space ( $z < 0$ ). Let the surface wave propagate

along  $x$ -axis  $\psi, \mathbf{u}_t \sim e^{iqx}$ . Due to the symmetry, the  $y$ -component of the vector  $\mathbf{u}_t$  equals to zero. Hence the solution of Eq. (90) can be taken as

$$\mathbf{u}_t = \mathbf{a} e^{\kappa z} e^{iqx} \quad (103)$$

where  $\mathbf{a} = (a_x, 0, a_z)$  is a two dimensional constant vector and

$$\kappa^2 = q^2 - \omega^2/c_t^2. \quad (104)$$

Due to the condition  $\nabla \mathbf{u}_t = 0$  (see Eq. (90)) the constants  $a_x$  and  $a_z$  are not independent:

$$a_x = \frac{\kappa}{-iq} a_z.$$

The general solution of Es.(89, 91) takes the form

$$\psi = (A e^{qz} + B e^{pz}) e^{iqx}, \quad (105)$$

$$\varphi_< = - \left( \frac{\omega^2 m}{e} A e^{qz} + 4\pi e n_0 B e^{pz} \right) e^{iqx}, \quad (106)$$

with

$$p^2 = q^2 + (\omega_p^2 - \omega^2)/c_l^2.$$

In the upper half-space we have the solution of the Poisson equation (92):

$$\varphi_> = C e^{-qz} e^{iqx}. \quad (107)$$

The relationship of the constants  $B$  and  $C$  to the constant  $A$  can be easily obtained from the electrostatic boundary conditions (94), (95). The result for the potential of the irrotational part of displacement  $\psi$  is

$$\psi = A \left[ e^{qz} + \left( 1 - \frac{\omega^2}{\omega_s^2} \right) e^{pz} \right] e^{iqx} \quad (108)$$

where  $\omega_s = \omega_p/\sqrt{2}$  is the classical surface plasmon frequency in the infinite wave length limit. Let us rewrite the boundary condition (93) explicitly in the case of the plane boundary :

$$(c_l^2 - 2c_t^2) n_t \nabla^2 \psi + 2c_t^2 \partial_z^2 \psi + 4c_t^2 \partial_z u_t^z = 0, \\ c_t^2 (\partial_x \partial_z \psi + \partial_x u_t^z + \partial_z u_t^x) = 0.$$

Substitution of the solutions (103) and (108) to the boundary conditions gives the system of equations for the frequencies of the surface plasma modes:

$$\left( 1 - \frac{\omega^2}{\omega_p^2} \right) (\omega^2 - \omega_s^2 - 2c_t^2 q^2) A - 2c_t^2 \kappa a_z = 0, \quad (109)$$

$$c_t^2 q^2 \left[ q + \left( 1 - \frac{\omega^2}{\omega_s^2} \right) p \right] A + c_t^2 (\kappa^2 + q^2) a_z = 0. \quad (110)$$

At  $q = 0$  we have

$$\omega^2 = \omega_s^2 \equiv \frac{\omega_p^2}{2}; \quad a_z = 0,$$

which corresponds to the usual infinite wave length surface plasma oscillations. At small  $q$  the dispersion of the surface plasmon takes the form

$$\omega^2 = \omega_s^2 + 2c_t^2 q^2 + i \frac{4c_t^3 q^3}{\omega_s}. \quad (111)$$

The damping of the surface plasmon is not surprising since at  $q \neq 0$  there is mixing with the transverse modes which can not decay from the surface at high frequency. There are decaying transverse solutions only for  $\omega^2 < c_t^2 q^2$ . Hence only the propagating transverse waves are allowed at the frequency  $\sim \omega_s$ . The coupling of the longitudinal and transverse components leads to the energy transfer to propagating waves and, consequently, provides damping of the surface plasmon. It should be mentioned that the hydrodynamic theory does not take into account the Landau damping due to the single particle excitations. Inclusion of the Landau damping will bring the usual contribution to the lifetime of the surface plasmon. Yet in a weakly coupled electron gas the Landau damping destroys the transverse modes. More precisely, if  $c_t < v_F$  the transverse modes are damped. Hence the decay channel of the surface plasmon into the transverse waves will be really open if the inverse inequality  $c_t > v_F$  is fulfilled and transverse modes are well defined. Using Eq. (97) for  $c_t$  and Eq. (77) for  $\mu$  we can express the condition  $c_t > v_F$  in terms of Landau Fermi liquid parameters:

$$F_1(1 + F_2/5) + 3F_2/5 > 6.$$

Consequently, the transverse modes and the decay channel described above could exist for strongly correlated charged Fermi liquid. It was mentioned in Ref. 15 that the condition should be satisfied for a electron liquid of low density.

With the accuracy up to  $q^2$  we can neglect the imaginary term in the dispersion law (111). In this long wavelength approximation the surface plasmon is a longitudinal wave with the dispersion

$$\omega^2 = \omega_s^2 + 2c_t^2 q^2. \quad (112)$$

The last equation shows that the dispersion of the almost longitudinal surface plasmon is totally determined by the shear modulus ( $c_t^2 \sim \mu$ ). Within the BHT the shear modulus equals to zero. In the case  $c_t = 0$  equations (109, 110) lead to the dispersionless surface plasmon with the frequency  $\omega = \omega_s$ . This agrees with the results obtained in<sup>7</sup> for a wide parabolic quantum well. The correct dispersion law (112) obtained from the generalized hydrodynamics has the dispersion coefficient  $2c_t^2$  which is of the same order of magnitude as the coefficient  $v_0^2 = c_l^2$  in the volume plasmon dispersion (80). For example, in the case of a high density electron gas ( $F_l \approx 0$ ) we have the ratio

$$2c_t^2/c_l^2 = \frac{2}{3}.$$

## VI. THE FOURTH-ORDER THEORY. THERMAL CONDUCTIVITY

As we have seen in Sec. III, the theory of the second order in parameter  $\gamma$  (26) does not contain effects of thermal conductivity which are responsible for the thermal equilibration processes. Hence the second-order approximation cannot give correct static solution which should correspond to a constant temperature ( $\nabla T(\mathbf{r}) = 0$ ). To account for the heat conduction one has to consider at least fourth-order approximation which contains two additional equations for the third  $\mathbf{L}^{(3)}$  and fourth  $\mathbf{L}^{(4)}$  moments. The system of equations follows Eqs. (22)-(25). The first two equations - the continuity equation and the equation for the velocity field, are the same as in the second-order theory (Eqs. (27) and (28) respectively). In the equations for the scalar  $P$  and traceless  $\boldsymbol{\pi}$  parts of the second moment we get additional terms which are proportional to the gradient of the  $\mathbf{L}^{(3)}$  (see Eq. 24):

$$D_t P + \frac{5}{3} P \nabla \cdot \mathbf{v} + \frac{2}{3} (\boldsymbol{\pi} \cdot \nabla) \cdot \mathbf{v} + Tr(\nabla \cdot \mathbf{L}^{(3)}) = 0, \quad (113)$$

$$D_t \boldsymbol{\pi} + \boldsymbol{\pi}(\nabla \cdot \mathbf{v}) + \{(\boldsymbol{\pi} \cdot \nabla) \otimes \mathbf{v}\}_S - \frac{2}{3} (\boldsymbol{\pi} \cdot \nabla) \cdot \mathbf{v} + P \left( \{\nabla \otimes \mathbf{v}\}_S - \frac{2}{3} \nabla \cdot \mathbf{v} \right) + \nabla \cdot \mathbf{L}^{(3)} - Tr(\nabla \cdot \mathbf{L}^{(3)}) = \mathbf{I}^{(2)}. \quad (114)$$

To close the set of equations with the accuracy of  $\gamma^4$  we should take an equation for  $\mathbf{L}^{(3)}$  in the general form (25) and omit the spatial derivatives of the fifth and the second moments in the equation for  $\mathbf{L}^{(4)}$ :

$$D_t \mathbf{L}^{(3)} + \mathbf{L}^{(3)}(\nabla \cdot \mathbf{v}) + \{(\mathbf{L}^{(3)} \cdot \nabla) \otimes \mathbf{v}\}_S + \nabla \cdot \mathbf{L}^{(4)} - \frac{1}{mn} \{\mathbf{L}^{(2)} \otimes (\nabla \cdot \mathbf{L}^{(2)})\}_S = \mathbf{I}^{(3)}, \quad (115)$$

$$D_t \mathbf{L}^{(4)} + \mathbf{L}^{(4)}(\nabla \cdot \mathbf{v}) + \{(\mathbf{L}^{(4)} \cdot \nabla) \otimes \mathbf{v}\}_S = \mathbf{I}^{(4)}. \quad (116)$$

Eqs. (27), (28) and (113)-(116) constitute the system of the generalized hydrodynamics equations in the fourth-order approximation.

The last term in the left-hand side of Eq. (113) can be rewritten as a divergency of a vector

$$Tr(\nabla \cdot \mathbf{L}^{(3)}) = \nabla_j L_{ij}^{(3)} = \nabla_j Q_j, \quad (117)$$

where the vector  $Q_j = L_{ij}^{(3)}$  describes the flow of energy in a comoving frame.

To demonstrate the recovering of the thermal conductivity we take the moments of the collision integral  $\mathbf{I}^{(3)}$  and  $\mathbf{I}^{(4)}$  in a linear approximation, similar to (36):

$$\mathbf{I}^{(3)} = -\nu_{c1} \mathbf{L}^{(3)}, \quad (118)$$

$$\mathbf{I}^{(4)} = -\nu_{c2} (\mathbf{L}^{(4)} - \mathbf{L}_F), \quad (119)$$

where we introduced the notation  $\mathbf{L}_F$  for the fourth moment of the distribution function  $f_{\mathbf{p}}^L(\mathbf{r}, t)$  (see Eqs. (31)-(34)):

$$\mathbf{L}_F = \frac{1}{m^3} \sum_{\mathbf{p}} p_i p_j p_k p_l f_{\mathbf{p}}^L \quad (120)$$

Due to the spherical symmetry of the function  $f_{\mathbf{p}}^L$  the fourth rank tensor  $\mathbf{L}_F$  (120) takes the form:

$$\mathbf{L}_F = (\delta_{ij}\delta_{kl} + \delta_{kj}\delta_{il} + \delta_{lj}\delta_{ik}) L_F, \quad (121)$$

$$L_F = \frac{1}{15} \sum_{\mathbf{p}} \frac{p^4}{m^3} f_{\mathbf{p}}^L.$$

Parameters  $\nu_{c1}$  and  $\nu_{c2}$  in Eqs. (118) and (119) can be considered as phenomenological coefficients which are, in general, not equal to the parameter  $\nu_c$  (36), but have the same order of magnitude.

In the low frequency limit  $\omega/\nu_c \ll 1$  we solve Eqs. (114)-(116) using a perturbation theory. Since the collision terms in all these equations have an order of magnitude of the collision frequency the solution in zero order of  $1/\nu_c$  follows from the equations

$$\mathbf{I}^{(j)} = 0, \quad j = 2, 3, 4$$

and takes the local equilibrium form

$$\mathbf{L}^{(4)} = \mathbf{L}_F, \quad \mathbf{L}^{(3)} = 0, \quad \boldsymbol{\pi} = 0.$$

The stress tensor becomes diagonal and proportional to the local pressure

$$\mathbf{L}^{(2)} = \mathbf{1} P_F$$

The first order solution of equation (30) for the tensor  $\boldsymbol{\pi}$  has been already obtained in Sec. III. It has the form of the viscosity tensor (41). Eq. (114), which determines  $\boldsymbol{\pi}$  with the accuracy of  $\gamma^4$ , differs from Eq. (30) since the third rank tensor  $\mathbf{L}^{(3)}$  contributes in the left-hand side of (114). However, the contribution of the third moment in the Eq. (114), being of the higher order in  $\omega/\nu_c$ , does not change the first order result of Sec. III. Hence the viscous solution (41) is still valid.

Let us consider solution of Eqs. (115), (116) for the third moment  $\mathbf{L}^{(3)}$ . In the first-order of  $\omega/\nu_c$  only the last two terms in the left-hand side in Eq. (115) contribute to the equation

$$-\nu_{c1} \mathbf{L}^{(3)} \approx \nabla \cdot \mathbf{L}^{(4)} - \frac{1}{mn} \{ \mathbf{L}^{(2)} \otimes (\nabla \cdot \mathbf{L}^{(2)}) \}_S, \quad (122)$$

where zero-order expressions for  $\mathbf{L}^{(4)}$  and  $\mathbf{L}^{(2)}$  should be used in the right-hand side.

Substituting zero-order (local equilibrium) solutions ( $\mathbf{L}^{(2)} = \mathbf{1} P_F$  and  $\mathbf{L}^{(4)} = \mathbf{L}_F$ ) to Eq. (122) one gets the first-order expression for the third moment

$$\mathbf{L}^{(3)} = -\frac{1}{\nu_{c1}} \left\{ \mathbf{1} \otimes \left( \nabla L_F - \frac{1}{mn} P_F \nabla P_F \right) \right\}_S. \quad (123)$$

Contraction of the tensor  $\mathbf{L}^{(3)}$  (123) over the couple of indexes gives the energy flow vector  $Q_j = L_{ij}^{(3)}$ , which enters the equation of the energy conservation (113)

$$\mathbf{Q} = -\frac{5}{\nu_{c1}} \left( \nabla L_F - \frac{1}{mn} P_F \nabla P_F \right). \quad (124)$$

Let us show that vector  $\mathbf{Q}$  is proportional to the gradient of temperature  $\nabla T(\mathbf{r}, t)$  at arbitrary degeneracy of an electron gas. The functions  $L_F$  and  $P_F$  depend on spatial coordinates via the spatial dependence of the local chemical potential  $\mu(\mathbf{r}, t)$  and the local temperature  $T(\mathbf{r}, t)$  entering the distribution function  $f_{\mathbf{p}}^L(\mathbf{r}, t)$

$$f_{\mathbf{p}}^L(\mathbf{r}, t) = f^L \left( \frac{\varepsilon_{\mathbf{p}} - \mu}{T} \right), \quad \varepsilon_{\mathbf{p}} = \frac{p^2}{2m}.$$

Hence gradients of  $P_F$  and  $L_F$  can be rewritten as follows

$$\nabla P_F = \frac{1}{3m} \sum_{\mathbf{p}} p^2 \nabla f_{\mathbf{p}}^L = \left( -\nabla \mu + \frac{\mu}{T} \nabla T \right) \frac{1}{3m} \sum_{\mathbf{p}} p^2 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} - \frac{\nabla T}{T} \frac{1}{6m^2} \sum_{\mathbf{p}} p^4 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} \quad (125)$$

$$\nabla L_F = \frac{1}{15m^3} \sum_{\mathbf{p}} p^4 \nabla f_{\mathbf{p}}^L = \left( -\nabla \mu + \frac{\mu}{T} \nabla T \right) \frac{1}{15m^3} \sum_{\mathbf{p}} p^2 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} - \frac{\nabla T}{T} \frac{1}{30m^4} \sum_{\mathbf{p}} p^6 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} \quad (126)$$

The partial integration allows one to express the momentum integrals in Eqs. (125), (126) in terms of macroscopic variables  $n$ ,  $P_F$  and  $L_F$ :

$$\begin{aligned}\frac{1}{3m} \sum_{\mathbf{p}} p^2 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} &= - \sum_{\mathbf{p}} f_{\mathbf{p}}^L = -n, \\ \sum_{\mathbf{p}} p^4 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} &= -5m \sum_{\mathbf{p}} p^2 f_{\mathbf{p}}^L = -15m^2 P_F, \\ \frac{1}{30m^4} \sum_{\mathbf{p}} p^6 \frac{\partial f_{\mathbf{p}}^L}{\partial \varepsilon_{\mathbf{p}}} &= -\frac{7}{30m^3} \sum_{\mathbf{p}} p^4 f_{\mathbf{p}}^L = -\frac{7}{2} L_F.\end{aligned}$$

Using the last expressions we can represent Eqs. (125) and (126) in the following compact form

$$\nabla P_F = \left( \nabla \mu - \frac{\mu}{T} \nabla T \right) n + \frac{5P_F}{2T} \nabla T, \quad (127)$$

$$\nabla L_F = \left( \nabla \mu - \frac{\mu}{T} \nabla T \right) \frac{P_F}{m} + \frac{7L_F}{2T} \nabla T. \quad (128)$$

Comparison of Eqs. (127) and (128) with Eq. (124) shows that the contribution of the gradient of the chemical potential exactly cancels in the equation for the energy flow vector. Thus the vector of the energy flow takes the usual form

$$\mathbf{Q} = -\lambda \nabla T \quad (129)$$

with the coefficient of heat conduction

$$\lambda = \frac{5}{2\nu_{c1}T} \left( 7L_F - \frac{5P_F^2}{mn} \right). \quad (130)$$

Finally Eq. (113) with  $\pi$  (30) and  $\mathbf{Q}$  (130) transforms to the common hydrodynamical equation of the energy conservation<sup>9</sup>

$$D_t P + \frac{5}{3} P \nabla \cdot \mathbf{v} + \frac{2}{3} (\pi \cdot \nabla) \cdot \mathbf{v} + \nabla \cdot \mathbf{Q} = 0, \quad (131)$$

which contain both the viscous and the heat conduction terms.

The formula (130) for the coefficient of heat conduction is valid at arbitrary degree of degeneracy of an electron gas. In the nondegenerate case  $P_F = nT$  and  $L_F = nT^2/m$ . Hence Eq. (130) transforms to the result:

$$\lambda = \frac{5nT}{\nu_{c1}m}.$$

In a degenerate gas at  $T = 0$  we have  $L_F = 5P_F^2/7mn$  hence the expression in the brackets in (130) cancels. At  $T/\varepsilon_F \ll 1$  the correction has the order of magnitude  $(T/\varepsilon_F)^2$  and the coefficient of heat conduction (130) is proportional to  $T/\nu_{c1}$ . Since the collision frequency in a degenerate Fermi system is a quadratic function of  $T$  the heat conduction diverges as  $1/T$  (see, for example, Ref. 19).

Thus, in the low frequency limit the generalized hydrodynamics equations (27), (28), (113)-(116) coincides with the correct set of the common hydrodynamics which consist of the continuity equation, Navier-Stokes equation and the equation of the energy conservation in the form (131).

In the high frequency (collisionless) limit the generalized hydrodynamics leads to the following plasmon dispersion

$$\omega^2 = \omega_p^2 + \langle v_{\mathbf{p}}^2 \rangle q^2 + \frac{\langle v_{\mathbf{p}}^4 \rangle - \langle v_{\mathbf{p}}^2 \rangle^2}{\omega_p^2} q^4$$

that is exactly the result of the linear response theory.

The system of equations (27), (28), (113)-(116) with the collision terms (36), (118), (119) constitute the closed set of equations and provides the smooth interpolation between high and low frequency regimes.

## VII. CONCLUSION

A description of dynamics of many-electron systems in terms of macroscopic collective variables, which is usually referred to as a hydrodynamical approach, provides a simple, physically transparent and powerful tool for studying spatially inhomogeneous problems. The common Bloch's hydrodynamics of an electron gas is based on the assumption of a local equilibrium and consequently shows a number of inconsistencies. We have shown that it is possible to construct an inherently consistent generalized hydrodynamics which correctly describes both the collisionless high-frequency limit and the collision dominated low-frequency regime, where the theory coincides with the standard Navier-Stokes hydrodynamics. The theory follows from the long-wavelength expansion of a kinetic equation and requires inclusion of new collective variables with a nontrivial tensor structure. We remind that only scalar (pressure and density) and vector (velocity) variables enter the common hydrodynamics.

The appearance of higher rank tensors is physically evident since the occupied region in a momentum space loses its spherical symmetry under a general (not locally equilibrium) evolution of the system. A need to describe a non-spherical isoenergetic surface inevitably requires tensor variables. As long as this surface is smooth it is possible to approximately describe it as a surface of a finite order and thus to take into account the tensor fields of the finite rank. We have actually shown above that the smoothness of the surface, which bounds the occupied region in the momentum space, is governed by the basic parameter (26).

In extension of the previous publication,<sup>13</sup> we presented a new general and more transparent derivation of the generalized hydrodynamics and showed how the closed set of equations of the standard Navier-Stokes hydrodynamics (including the heat transport) is recovered

in the low frequency regime. We have shown that the generalized hydrodynamics can be built on the basis of Landau theory of Fermi liquid which allows to determine the correlation contribution to the stress tensor. Although this hydrodynamics (as the Landau theory of Fermi liquid itself) is meaningful only in the linear approximation, we believe that it could be of a practical importance since it allows to express the results of hydrodynamical calculations, which are relatively simple, in terms of the microscopic Landau parameters. For example, the theory offers a possibility to determine the contribution of Fermi-liquid correlation effects to the eigenmodes of spatially inhomogeneous systems.

In a linear approximation the tensor variables, which describe the absence of a local equilibrium, can be interpreted as an effective shear stress of a liquid. On a time scale much longer than the collision time the contribution of the shear stress vanishes and the dynamics of a liquid is governed only by the usual bulk stress. However, as the collective modes of a charged liquid normally belong to the high frequency range, the shear contribution cannot be neglected. It is important to realize that this contribution cannot be modeled, even qualitatively, merely by a change of the bulk modulus - the procedure, which has been frequently used in literature to obtain the correct dispersion of plasmon.<sup>3,5,7</sup> The reason is that the relative contribution of the shear and bulk stresses is different for different modes and different geometries. As a result the eigenfrequencies and the structure of plasma modes in confined systems strongly deviate from predictions of BHT even with "improved" bulk modulus, which is adjusted to provide a correct plasmon dispersion in a homogeneous situation. To demonstrate this qualitative non-local-equilibrium effect we calculated eigenmodes of a Fermi liquid confined by a harmonic potential of different dimensionality. The results show a nontrivial contribution of the shear modulus to eigenfrequencies of the longitudinal plasma oscillation. The most transparent result concerns surface waves at the edge of 1D harmonically trapped system. In the absence of the shear modulus these modes are absolutely dispersionless in agreement with the BHT results.<sup>7</sup> They acquire a dispersion of the same order of magnitude as the bulk plasmon due to deviation from the local equilibrium. We have also found that existence of a nonzero shear modulus may open a new channel of decay of these surface modes, in addition to the usual Landau damping.

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